PROOF OF THE BIEBERBACH CONJECTURE FOR A CERTAIN CLASS OF UNIVALENT FUNCTIONS

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ABSTRACT
In the following we prove that for a given univalent function such that $|a_2| < 0.867$, $|a_n| \leqq n$ for each n. The method of proof is closely related to Milin's method.

Let S be the class of univalent functions in $|z| < 1$ with the normalization

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

We denote

$$\log \frac{f(z)}{z} = 2 \sum_{k=1}^{\infty} \gamma_k z^k, f(z) \in S. \tag{2}$$

Milin [1] proved that

$$\sum_{k=1}^{n} k |\gamma_k|^2 \leqq \frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{k} \rho^{2k} + \log \frac{1}{1-r^2} \right), \ \rho = \frac{1}{r} > 1. \tag{3}$$

For $\rho = 2^{1/(2n+1)}$ Milin derived

$$\sum_{k=1}^{n} k |y_k|^2 \leqq \frac{1}{2}\left(\sum_{k=1}^{n} \frac{\rho^{2k}}{k} + \log \frac{1}{1-r^2} \right) \leqq \sum_{k=1}^{n} \frac{1}{k} + \delta \tag{4}$$

where $\delta < 0.312$.

Our aim is now to estimate $\sum_{k=2}^{n} k|\gamma_k|^2$ instead of $\sum_{k=1}^{n} k|\gamma_k|^2$. In exactly the same method as in [1], we derive

$$\sum_{k=2}^{n} k |\gamma_k|^2 \leqq \frac{1}{2}\left(\sum_{k=2}^{n} \frac{\rho^{2k}}{k} + \log \frac{1}{1-r^2} \right), \ \rho = \frac{1}{r} > 1. \tag{5}$$

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For $\rho = 2^{1/(2n+1)}$ we have:

$$\frac{1}{2}\left(\sum_{k=2}^{n} \frac{\rho^{2k}}{k} + \log\frac{1}{1-r^2}\right) = -\frac{\rho^2}{2} + \frac{1}{2}\left(\sum_{k=1}^{n} \frac{\rho^{2k}}{k} + \log\frac{1}{1-r^2}\right) < -\frac{1}{2} + \sum_{k=1}^{n} \frac{1}{k} + \delta.$$

We thus have from (5)

$$\sum_{k=2}^{n} k\left|\gamma_k\right|^2 \leqq \sum_{k=1}^{n} \frac{1}{k} + \delta - \frac{1}{2}. \tag{6}$$

Since $\gamma_1 = \dfrac{a_2}{2}$, we have

$$\sum_{k=1}^{n} k\left|\gamma_k\right|^2 \leqq \sum_{k=1}^{n} \frac{1}{k} + \delta - \frac{1}{2} + \left|\frac{a_2}{2}\right|^2. \tag{7}$$

We can now formulate our result:

THEOREM. *Let $f(z) \in S$ and $\left|a_2\right| < 0.867$. Then $\left|a_n\right| \leqq n$ for each n.*

PROOF. We have $\left|\dfrac{a_2}{2}\right|^2 + \delta - \dfrac{1}{2} < \left(\dfrac{0.867}{2}\right)^2 + 0.312 - \dfrac{1}{2} < 0.$

So we have from (7)

$$\sum_{k=1}^{n} k\left|\gamma_k\right|^2 < \sum_{k=1}^{n} \frac{1}{k} \text{ if } \left|a_2\right| < 0.867. \tag{8}$$

For the coefficients $\{b_n\}$ of the function $\sqrt{f(z^2)}$ we have

$$\left|b_n\right|^2 \leqq \exp\left(\sum_{k=1}^{n-1} k\left|\gamma_k\right|^2 - \sum_{k=1}^{n-1} \frac{1}{k}\right). \tag{9}$$

From (8) and (9), it follows that the coefficients $\{b_n\}$ are bounded by 1 for our class.

Our theorem follows readily by Robertson's procedure.

REMARK. As it is well known $\left|b_n\right| \leqq 1$ is not true in general, so the method can not yield the global result.

REFERENCE

1. Milin, I. M., *On the coefficients of univalent functions*, Soviet Math. Dokl., **8** (1967), 1255–1258.